WEIGHTED BOUND FOR COMMUTATORS

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ABSTRACT. Let K be the Calderón-Zygmund convolution kernel on $\mathbb{R}^d (d \geq 2)$. Define the commutator associated with K and $a \in L^\infty(\mathbb{R}^d)$ by

$$T_a f(x) = p.v. \int K(x-y) m_{x,y} a \cdot f(y) dy.$$

Recently, Grafakos and Honzík [5] proved that T_a is of weak type (1,1) for $d = 2$. In this paper, we show that T_a is also weighted weak type (1,1) with the weight $|x|^\alpha$ $(-2 < \alpha < 0)$ for $d = 2$. Moreover, we prove that T_a is bounded on weighted $L^p(\mathbb{R}^d)$ $(1 < p < \infty)$ for all $d \geq 2$.

1. Introduction

Suppose that K is the Calderón-Zygmund convolution kernel on $\mathbb{R}^d \setminus \{0\}$ $(d \geq 2)$, which means that K satisfies following three conditions:

$$|K(x)| \leq C|x|^{-d}, \tag{1.1}$$

$$\int_{R<|x|<2R} K(x) dx = 0, \text{ for all } R > 0, \tag{1.2}$$

$$|\nabla K(x)| \leq \frac{C}{|x|^{d+1}}. \tag{1.3}$$

In 1987, Christ and Journé [2] introduced a commutator associated with K and $a \in L^\infty(\mathbb{R}^d)$ by

$$T_a f(x) = \text{p.v.} \int K(x-y) m_{x,y} a \cdot f(y) dy, \qquad f \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class and

$$m_{x,y} a = \int_0^1 a((1-t)x + ty) dt.$$

Note that when $d = 1$, then

$$m_{x,y} a = \frac{\int_0^x a(z)dz - \int_0^y a(z)dz}{x-y}.$$

In this case, let $K(x) = \frac{1}{x}$ and $A(x) = \int_0^x a(z)dz$, then $A'(x) = a(x) \in L^\infty(\mathbb{R})$. So

$$T_a f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{A(x) - A(y)}{x-y} \frac{f(y)}{x-y} dy,$$

Date: October 5, 2013.

2010 *Mathematics Subject Classification.* 42B20.

Key words and phrases. weighted, weak type (1,1), Calderón commutator.

The work is supported by NSFC (No. 11371057) and SRFDP (No. 20130003110003).

which is the famous Calderón commutator discussed in [1].

In [2], Christ and Journé showed that T_a is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. In 1995, Hofmann [7] gave the weighted $L^p(\mathbb{R}^d)\,(1 < p < \infty)$ boundedness of T_a when the kernel $K(x) = \Omega(x/|x|)|x|^{-d}$. Recently, Grafakos and Honzík [5] proved that T_a is weak type $(1, 1)$ for $d = 2$. Further, Seeger [9] showed that T_a is still weak type $(1, 1)$ for all $d \geq 2$. The purpose of this paper is to establish a weighted variety of Grafakos and Honzík's results in [5]. In the sequel, for $1 \leq p \leq \infty$, A_p denotes the Muckenhoupt weight class and $L^p(\omega)$ denotes the weighted $L^p(\mathbb{R}^d)$ space with norm $\|\cdot\|_{p,\omega}$. We also denote $\omega(E) = \int_E \omega(x)dx$ for a measurable set E in $\mathbb{R}^d$. The main result obtained in the present paper is as follows.

Theorem 1.1. *Suppose K satisfies* (1.1), (1.2) *and* (1.3) *for $d = 2$. Let $a \in L^\infty(\mathbb{R}^2)$ and $\omega(x) = |x|^\alpha$ for $-2 < \alpha < 0$. Then there exists a constant $C > 0$ such that*

$$\omega(\{x \in \mathbb{R}^2 : |T_a f(x)| > \lambda\}) \leq C\lambda^{-1}\|a\|_\infty \|f\|_{1,\omega}$$

for all $\lambda > 0$ and $f \in L^1(\omega)$.

We would like to point out that the proof of Theorem 1.1 follows the nice idea from [5]. However, there are some differences in proving T_a is of weak type $(1, 1)$ for the weighted case. In fact, the essential difficulties of proving Theorem 1.1 are to show the smoothness of kernels of $(T_j^* T_j)_\omega$ and $(T_i^* T_j)_\omega$ (see (3.2) and (3.13) below, respectively), these estimates are more complicated than no weight case, although we only consider power weight $|x|^\alpha$ for $-2 < \alpha \leq 0$. Our main innovations are further decomposition of power weight according to the dyadic decomposition. Note that $|x|^\alpha \in A_1(\mathbb{R}^2)$ if and only if $-2 < \alpha \leq 0$, but our method cannot be used to deal with the general A_1 weight. This is the reason why we now cannot get a similar result as Theorem 1.1 for general weight $w \in A_1(\mathbb{R}^2)$.

In order to prove Theorem 1.1, we need to establish the weighted L^p boundedness of T_a (actually we only need weighted L^2 boundedness). Although the $L^p(\omega)$ boundedness of T_a given by [7, Theorem 2.15] for the homogeneous kernel $K(x) = \Omega(x/|x|)|x|^{-d}$, it seems that one cannot apply directly to T_a with the kernel satisfying (1.1)-(1.3) discussed in this paper. However, Hofmann established a weighted L^p boundedness criteria in [7] which is similar to $T1$ theorem. The proof of Theorem 1.2 given here is an application of that criteria. More precisely, T_a is a special example of the general operators studied in [7].

Theorem 1.2. *Suppose K satisfies the conditions* (1.1), (1.2) *and* (1.3)*. Let $a \in L^\infty(\mathbb{R}^d)\,(d \geq 2)$ and $\omega \in A_p$, $1 < p < \infty$. Then there exists a constant $C > 0$ such that*

$$\|T_a f\|_{p,\omega} \leq C\|a\|_\infty \|f\|_{p,\omega}. \tag{1.4}$$

This paper is organized as follows. The proof of Theorem 1.2 is given in Section 4. In Section 2, we complete the proof of Theorem 1.1 based on Theorem 1.2 and Lemma 2.3. Moreover, in this section, we also state that the proof of Lemma 2.3 can be reduced to two key lemmas, their proofs will be given in Section 3. Throughout this paper the letter C will stand for a positive

constant which is independent of the essential variables and not necessarily the same one in each occurrence.

2. Proof of Theorem 1.1

Let us begin by giving an analogous Calderón-Zygmund decomposition of $f \in L^1(\omega)$. First, we recall the Whitney decomposition which can be found in [3]:

Lemma 2.1. (**Whitney decomposition**) *Let F be an open nonempty proper subset of $\mathbb{R}^d$. Then there exists a family of dyadic closed cubes $\{Q_j\}_j$ such that*

(a) $\bigcup Q_j = F$ *and Q_j's have disjoint interior.*

(b) $\sqrt{d} \cdot l(Q_j) \le dist(Q_j, F^c) \le 4\sqrt{d} \cdot l(Q_j)$, *where $l(Q_j)$ denotes the side's length of Q_j.*

Lemma 2.2. *Let $\omega \in A_1$ and $f \in L^1(\omega)$. Set $E := \{Mf(x) > \frac{\lambda}{\|a\|_\infty}\}$ where M is the Hardy-Littlewood maximal operator. Then for $a \in L^\infty(\mathbb{R}^d)$ and $\lambda > 0$, we have the following conclusions:*

(i) $E = \bigcup\limits_n Q_n$, *Q_n's are disjoint dyadic cubes.*

(ii) $\omega(E) \le C\frac{\|a\|_\infty}{\lambda}\|f\|_{1,\omega}$.

(iii) $f = g + b$.

(iv) $b = \sum b_n$, $\mathrm{supp} b_n \subset Q_n$, $\int b_n = 0$, $\|b_n\|_1 \le C\frac{\lambda}{\|a\|_\infty}|Q_n|$, $\|b\|_{1,\omega} \le C\|f\|_{1,\omega}$.

(v) $\|g\|_{2,\omega}^2 \le C\frac{\lambda}{\|a\|_\infty}\|f\|_{1,\omega}$.

Proof. Since E is open, we can make a dyadic Whitney decomposition of the set E. Thus E is the union of the disjoint dyadic cubes Q_n and we have

$$\sqrt{d}\, l(Q_n) < dist(Q_n, E^c) < 4\sqrt{d}\, l(Q_n). \tag{2.1}$$

By the weighted weak type (1,1) of M, we have

$$\omega(E) \le C\frac{\|a\|_\infty}{\lambda}\|f\|_{1,\omega}. \tag{2.2}$$

We write $f = g + b$, where $g = f\chi_E^c + \sum\limits_n \frac{1}{|Q_n|}\int_{Q_n} f(x)dx\chi_{Q_n}$, $b = \sum\limits_n \{f - \frac{1}{|Q_n|}\int_{Q_n} f(x)dx\}\chi_{Q_n} =: \sum\limits_n b_n$. So, b_n supports in Q_n and $\int b_n = 0$. Let tQ_n denote the cube with t times the side length of Q_n and the same center. We first claim that

$$\frac{1}{|Q_n|}\int_{Q_n} |f(x)|dx \le C\frac{\lambda}{\|a\|_\infty}. \tag{2.3}$$

In fact, by the Whitney decomposition's property (2.1) we have $9\sqrt{d}Q_n \cap E^c \ne \emptyset$. Thus by the definition of E, there exists $x_0 \in 9\sqrt{d}Q_n$ such that $Mf(x_0) \le \frac{\lambda}{\|a\|_\infty}$. Using the property of maximal function, we have $\frac{1}{|9\sqrt{d}Q_n|}\int_{9\sqrt{d}Q_n} |f(x)|dx \le C\frac{\lambda}{\|a\|_\infty}$. Hence we have the estimate

$$\frac{1}{|Q_n|}\int_{Q_n} |f(x)|dx \le \frac{1}{|Q_n|}\int_{9\sqrt{d}Q_n} |f(x)|dx \le C\frac{\lambda}{\|a\|_\infty}.$$

For b_n and b, by (2.2) and (2.3) we have

$$\|b_n\|_1 \le 2\int_{Q_n} |f(x)|dx \le C\frac{\lambda}{\|a\|_\infty}|Q_n|,$$

$$\|b\|_{1,\omega} \le \|f\|_{1,\omega} + C\frac{\lambda}{\|a\|_\infty}\omega(E) \le C\|f\|_{1,\omega}.$$

Note that if $x \in E^c$, it is obvious that $|f(x)| \le \frac{\lambda}{\|a\|_\infty}$. Using this fact, (2.2) and (2.3), we have

$$\|g\|_{2,\omega}^2 \le \frac{\lambda}{\|a\|_\infty}\|f\|_{1,\omega} + C\big(\frac{\lambda}{\|a\|_\infty}\big)^2\omega(E) \le C\frac{\lambda}{\|a\|_\infty}\|f\|_{1,\omega}.$$

$\square$

In the following we use Lemma 2.2 for $d = 2$ and $\omega(x) = |x|^\alpha$ with $-2 < \alpha < 0$. Denote $\mathfrak{Q}_k = \{Q_n : l(Q_n) = 2^k\}$ and let $B_k = \sum_{Q\in\mathfrak{Q}_k} b_Q$. Taking a smooth function ϕ on $[0,\infty)$ such that supp$\phi \subset \{x : \frac{1}{4} \le |x| \le 1\}$ and $\sum_j \phi_j(x) = 1$ for all $x \in \mathbb{R}^2\backslash\{0\}$, where $\phi_j(x) = \phi(2^{-j}x)$. Write $K = \sum_j K_j$, where $K_j(x) = \phi_j(x)K(x)$ and define the corresponding operators T_j with the kernel $K_j(x-y)m_{x,y}a$. Clearly we have $T_a = \sum_j T_j$.

We now state a lemma, which plays an important role in the proof of Theorem 1.1:

Lemma 2.3. *There exists an $\varepsilon > 0$ such that for any integer $s \ge 10$,*

$$\Big\|\sum_j T_j B_{j-s}\Big\|_{2,\omega}^2 \le C2^{-\varepsilon s}\lambda\|a\|_\infty\|b\|_{1,\omega}, \tag{2.4}$$

where C is a constant depended on K only.

The proof of Lemma 2.3 will be stated below. We now explain that Theorem 1.1 can be obtained by Lemma 2.3 and Theorem 1.2. In fact, for any $f \in L^1(\omega)$ and $\lambda > 0$, by Lemma 2.2, we have

$$\omega(\{|T_af(x)| > \lambda\}) \le \omega\big(\{|T_ag(x)| > \lambda/2\}\big) + \omega\big(\{|T_ab(x)| > \lambda/2\}\big).$$

Since $g \in L^2(\omega)$, by Theorem 1.2, we have $\|T_ag\|_{2,\omega} \le C\|a\|_\infty\|g\|_{2,\omega}$. Hence, by Chebychev's inequality and Lemma 2.2,

$$\omega(\{|T_ag(x)| > \lambda/2\}) \le 4\|T_ag\|_{2,\omega}^2/\lambda^2 \le C\frac{\|a\|_\infty^2\lambda\|f\|_{1,\omega}}{\|a\|_\infty\lambda^2} = C\|a\|_\infty\frac{\|f\|_{1,\omega}}{\lambda}.$$

Let $E^* = \bigcup 2^{11}Q_n$. Then we have

$$\omega(\{|T_ab(x)| > \lambda/2\}) \le \omega(E^*) + \omega(\{x \in (E^*)^c : |T_ab(x)| > \lambda/2\}).$$

Since ω satisfies the doubling condition, the set E^* satisfies

$$\omega(E^*) \le C\omega(E) \le C\frac{\|a\|_\infty}{\lambda}\|f\|_{1,\omega}. \tag{2.5}$$

We write

$$T_ab(x) = \sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} T_jB_{j-s}.$$

Note that $T_jB_{j-s}(x)=0$, for $x\in(E^*)^c$ and $s<10$. Therefore

$$\omega\Big(\Big\{x\in(E^*)^c: T_ab(x)>\frac{\lambda}{2}\Big\}\Big)=\omega\Big(\Big\{x\in(E^*)^c:\Big|\sum_{s\geq 10}\sum_{j\in\mathbb{Z}}T_jB_{j-s}(x)\Big|>\frac{\lambda}{2}\Big\}\Big).$$

From Lemma 2.3 we get

$$\|\sum_{s\geq 10}\sum_{j\in\mathbb{Z}}T_jB_{j-s}\|_{2,\omega}^2\leq\big(\sum_{s\geq 10}\|\sum_{j\in\mathbb{Z}}T_jB_{j-s}\|_{2,\omega}\big)^2\leq C\lambda\|a\|\|_\infty\|b\|_{1,\omega}.$$

By Chebychev's inequality, we have

$$\omega\Big(\Big\{x\in(E^*)^c:\Big|\sum_{s\geq 10}\sum_{j\in\mathbb{Z}}T_jB_{j-s}(x)\Big|>\frac{\lambda}{2}\Big\}\Big)\leq C\|a\|_\infty\frac{\|b\|_{1,\omega}}{\lambda}\leq C\|a\|_\infty\frac{\|f\|_{1,\omega}}{\lambda}.$$

Hence we get the conclusion of Theorem 1.1. Thus, to complete the proof of Theorem 1.1, it suffices to show Lemma 2.3 and Theorem 1.2. The proof of Theorem 1.2 will be given in Section 4. Let us first state Lemma 2.3. We write

$$\begin{aligned}&\|\sum_{j\in\mathbb{Z}}T_jB_{j-s}\|_{2,\omega}^2=\sum_{i,j\in\mathbb{Z}}\langle T_jB_{j-s},T_iB_{i-s}\rangle_\omega\\&=\sum_{j\in\mathbb{Z}}\|T_jB_{j-s}\|_{2,\omega}^2+2\sum_i\sum_{j=i-2}^{i-1}\langle T_jB_{j-s},T_iB_{i-s}\rangle_\omega+2\sum_{i\in\mathbb{Z}}\sum_{j\leq i-3}\langle T_jB_{j-s},T_iB_{i-s}\rangle_\omega\\&=:I+II+III,\end{aligned}$$

where $\langle u,v\rangle_\omega=\int u(x)v(x)\omega(x)dx$ for the real valued functions u and v.

Note that the estimate of II can be reduced to I:

$$\begin{aligned}2\Big|\sum_i\sum_{j=i-2}^{i-1}\langle T_jB_{j-s},T_iB_{i-s}\rangle_\omega\Big|&\leq\sum_i\sum_{j=i-2}^{i-1}(\|T_jB_{j-s}\|_{2,\omega}^2+\|T_iB_{i-s}\|_{2,\omega}^2)\\&\leq 4\sum_i\|T_iB_{i-s}\|_{2,\omega}^2.\end{aligned}$$

Hence, if we can establish the following lemma, then we may get the estimate of I and II.

Lemma 2.4. *There exists an $\varepsilon>0$ such that for any fixed $s\geq 10$,*

$$\|T_jB_{j-s}\|_{2,\omega}^2\leq C2^{-\varepsilon s}\lambda\|a\|_\infty\|B_{j-s}\|_{1,\omega},\tag{2.6}$$

where C is a constant dependent on the properties of K.

To handle the cross terms III, we need the following conclusion:

Lemma 2.5. *There exist C , $\varepsilon>0$ such that*

$$\Big|\sum_{i\in\mathbb{Z}}\sum_{j\leq i-3}\langle T_jB_{j-s},T_iB_{i-s}\rangle_\omega\Big|\leq C2^{-\varepsilon s}\lambda\|a\|_\infty\|b\|_{1,\omega}$$

for any $s\geq 10$.

So, to get Lemma 2.3, it remains to prove Lemma 2.4 and Lemma 2.5, which will be given in the following section.

3. Proofs of lemma 2.4 and lemma 2.5

3.1 Proof of Lemma 2.4

First let us consider Lemma 2.4. For any $i, j \in \mathbb{Z}$, we write

$$\langle T_j B_{j-s}, T_i B_{i-s} \rangle_\omega = \langle (T_i^* T_j)_\omega B_{j-s}, B_{i-s} \rangle,$$

where $(T_i^* T_j)_\omega$ has the kernel

$$K_{i,j}(y,x) = \int K_i(z-y) K_j(z-x) m_{x,z}a \cdot m_{y,z}a \cdot \omega(z) dz. \tag{3.1}$$

Hence we can write

$$\|T_j B_{j-s}\|_{2,\omega}^2 = \langle (T_j^* T_j)_\omega B_{j-s}, B_{j-s} \rangle,$$

It is easy to see that the following two lemmas are the key to proving Lemma 2.4.

Lemma 3.1. *For $|y| > 2^{j+1}$ or $|y| < 2^{j-3}$, there exist C , $\varepsilon > 0$ such that*

$$|(T_j^* T_j)_\omega B_{j-s}(y)| \le C 2^{-\varepsilon s} \lambda \|a\|_\infty \omega(y)$$

for any integer $s \ge 10$.

Lemma 3.2. *For $2^{j-3} \le |y| \le 2^{j+1}$, there exist C , $\varepsilon > 0$ such that*

$$|(T_j^* T_j)_\omega B_{j-s}(y)| \le C 2^{-\varepsilon s} \lambda \|a\|_\infty \omega(y)$$

for any integer $s \ge 10$.

Proof of Lemma 3.1: We claim that the kernel $K_{j,j}$ which is given by (3.1) has the Hölder smoothness:

$$|K_{j,j}(y,x) - K_{j,j}(y,x')| \le C 2^{\frac{19}{20}j} |x-x'|^{\frac{1}{20}} 2^{-3j} \|a\|_\infty^2 \omega(y), \tag{3.2}$$

for any $|x-x'| \le 2^{j-10}$. Once we establish (3.2), we can get Lemma 3.1. In fact, write

$$\begin{aligned}
\Big| \int K_{j,j}(y,x) B_{j-s}(x) dx \Big| &= \Big| \sum_{Q_n \in \mathfrak{Q}_{j-s}} \int K_{j,j}(y,x) b_n(x) dx \Big| \\
\le \Big| \sum_{Q_n \in \mathfrak{Q}_{j-s}} & \int_{|y-x| > 10 \cdot 2^{\frac{9}{10}j} |x - x_{Q_n}|^{\frac{1}{10}}} (K_{j,j}(y,x) - K_{j,j}(y, x_{Q_n})) b_n(x) dx \Big| \\
+ \Big| \sum_{Q_n \in \mathfrak{Q}_{j-s}} & \int_{|y-x| < 10 \cdot 2^{\frac{9}{10}j} |x - x_{Q_n}|^{\frac{1}{10}}} (K_{j,j}(y,x) - K_{j,j}(y, x_{Q_n})) b_n(x) dx \Big| \\
=: J_1 + J_2, &
\end{aligned}$$

where x_{Q_n} denotes the center of Q_n. For J_1, by using the Hölder smoothness (3.2) we have

$$\begin{aligned}
J_1 &\le \sum_{Q_n \in \mathfrak{Q}_{j-s}} c 2^{\frac{19}{20}j - 3j} \|a\|_\infty^2 \omega(y) \int |x - x_{Q_n}|^{\frac{1}{20}} |b_n(x)| dx \\
&\le C 2^{-2j} 2^{-\frac{s}{20}} \omega(y) \lambda \|a\|_\infty \sum_{dist(Q_n, y) \le 2^{j+1}} |Q_n| \\
&\le C 2^{-\frac{s}{20}} \omega(y) \lambda \|a\|_\infty.
\end{aligned}$$

For J_2, we have

$$J_2 \le C2^{-2j}\omega(y)\lambda\|a\|_\infty \sum_{dist(Q_n,y)\le 10\cdot 2^{j-\frac{s}{10}}} |Q_n| \le C2^{-\frac{s}{5}}\|a\|_\infty\lambda\omega(y).$$

Then we can choose $\varepsilon = \frac{1}{20}$. To obtain the kernel's Hölder smoothness (3.2), write

$$K_{j,j}(y,x) - K_{j,j}(y,x') =: A_1 + A_2,$$

where

$$A_1 = \int (K_j(z-x) - K_j(z-x'))K_j(z-y)m_{x,z}a \cdot m_{y,z}a \cdot \omega(z)dz$$

and

$$A_2 = \int K_j(z-x')K_j(z-y)(m_{x,z}a - m_{x',z}a)m_{y,z}a \cdot \omega(z)dz.$$

Note that $2^{j-2} \le |z-y| \le 2^j$. Then $|z| \ge C \cdot \max\{|y|, 2^{j-3}\}$ when $|y| < 2^{j-3}$ or $|y| > 2^{j+1}$. Thus, $\omega(z) \le C\omega(y)$. Since K_j is a smooth function with compact support, we have

$$|K_j(z-x) - K_j(z-x')| = \left| \int_0^1 \langle x'-x, \nabla K_j(z-(1-s)x'-sx)\rangle ds \right| \le C2^{-3j}|x-x'|.$$

Therefore

$$|A_1| \le C2^{-3j}\|a\|_\infty^2\omega(y)|x-x'|. \tag{3.3}$$

To estimate A_2, we switch to polar coordinates $z = y + r\theta$, then

$$A_2 = \int_{\mathbb{S}^1}\int_{2^{j-2}}^{2^j} \psi(r)(m_{x,y+r\theta}a - m_{x',y+r\theta}a)\omega(y+r\theta)drd\theta, \tag{3.4}$$

where

$$\psi(r) = K_j(y-x'+r\theta)K_j(r\theta)m_{y,y+r\theta}a \cdot r = K_j(y-x'+r\theta)K_j(r\theta)\int_0^r a(y+s\theta)ds.$$

It is easy to see that $\|\psi\|_\infty \le C2^{-3j}\|a\|_\infty$ and $\|\psi'\|_\infty \le C2^{-4j}\|a\|_\infty$.

We first split the integral over $\mathbb{S}^1$ as a sum over the arc

$$\left|\theta \pm \frac{x-y}{|x-y|}\right| < t_0$$

and its complement, t_0 will be chosen later as $C2^{-\frac{1}{10}j}|x-x'|^{\frac{1}{10}}$. Therefore the part of the integral in (3.4) over this arc is bounded by $C2^{-2j}\|a\|_\infty^2 t_0\omega(y)$.

Now we reduce A_2 to estimate the part of the outer integral in (3.4) over the set

$$\left|\theta \pm \frac{x-y}{|x-y|}\right| \ge t_0.$$

By a rotation, without loss of generality, we can assume that $\theta = (1,0)$,

$$\begin{aligned} N = &\int_{2^{j-2}}^{2^j} \psi(r)\int_0^1 a(x+s((r,0)-x+y))\omega(y+r(1,0))dsdr \\ &- \int_{2^{j-2}}^{2^j} \psi(r')\int_0^1 a(x'+s((r',0)-x'+y))\omega(y+r'(1,0))dsdr'. \end{aligned}$$

We make a coordinate transform. For the first term in the above integral, set

$$u = x_1 + s(r - x_1 + y_1) \quad \text{and} \quad v = x_2 + s(-x_2 + y_2),$$

then $r = \frac{u-x_1}{v-x_2}(y_2 - x_2) - y_1 + x_1$. For the second term, let

$$u = x'_1 + s(r' - x'_1 + y_1) \quad \text{and} \quad v = x_2 + s(-x'_2 + y_2),$$

then $r' = \frac{u-x'_1}{v-x'_2}(y_2 - x'_2) - y_1 + x'_1$. Therefore, we have

$$N = \iint_A \psi(r)a(u,v)\omega(y + r(1,0))\frac{dudv}{|x_2 - v|} - \iint_{A'} \psi(r')a(u,v)\omega(y + r'(1,0))\frac{dudv}{|x'_2 - v|},$$

where A is the triangle with vertices $\{y + (2^{j-2}, 0), y + (2^j, 0), x\}$ and A' is the triangle with vertices $\{y + (2^{j-2}, 0), y + (2^j, 0), x'\}$. By symmetric, we may assume $x_2 > y_2$. Observe that

$$\iint_A \frac{dudv}{|x_2 - v|} = \int_{y_2}^{x_2} \frac{3 \cdot 2^j}{4} \frac{x_2 - v}{x_2 - y_2} \frac{dv}{x_2 - v} = \frac{3}{4} 2^j.$$

Now we assume that

$$|x - x'| \leq 2^{j-10} \quad \text{and} \quad |x - y| > 10|x - x'|^{\frac{1}{10}} 2^{\frac{9}{10}j}$$

and set $t_0 = 10|x - x'|^{\frac{1}{10}} 2^{-\frac{1}{10}j}$. Since $\left|(1,0) \pm \frac{x-y}{|x-y|}\right| \geq t_0$ and t_0 is small relative to 2^j, we have

$$\frac{|x_2 - y_2|}{|x - y|} > \frac{1}{10} t_0 = |x - x'|^{\frac{1}{10}} 2^{-\frac{1}{10}j}.$$

Then $|x_2 - y_2| \geq 10|x - x'|^{\frac{1}{5}} 2^{\frac{4}{5}j}$. By using an analogous method we obtain $|x'_2 - y_2| \geq 9|x - x'|^{\frac{1}{5}} 2^{\frac{4}{5}j}$. Using polar coordinate transform we get

$$\begin{aligned} \Big| \iint_{A \bigcap B(x, 2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}})} \frac{dudv}{|x_2 - v|} \Big| &\leq \iint_0^{2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}}} \frac{drd\theta}{|\sin\theta|} \\ &\leq C 2^{\frac{3}{4}j}|x - x'|^{\frac{1}{4}} \frac{2^j}{|x_2 - y_2|} \leq C 2^{\frac{19}{20}j}|x - x'|^{\frac{1}{20}}, \end{aligned}$$

where the angle θ is between vector $v - x$ and $(1,0)$ and the second inequality comes from the geometry estimate $|\sin\theta| \geq C\frac{x_2 - y_2}{2^j}$. So we have

$$\Big| \iint\limits_{A \bigcap B(x, 2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}})} \psi(r)a(u,v)\omega(y + (r,0))\frac{dudv}{|x_2 - v|} \Big| \leq C 2^{-3j}\|a\|_\infty^2 \omega(y) 2^{\frac{19}{20}j}|x - x'|^{\frac{1}{20}}. \tag{3.5}$$

Since $A' \bigcap B(x, 2^{\frac{3}{4}j}|x - x'|^{\frac{1}{4}}) \subseteq A' \bigcap B(x', 2 \cdot 2^{\frac{3}{4}j}|x - x'|^{\frac{1}{4}})$, by an analogous method we also have the estimate

$$\Big| \iint\limits_{A' \bigcap B(x, 2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}})} \psi(r')a(u,v)\omega(y + (r',0))\frac{dudv}{|x_2 - v|} \Big| \leq C 2^{-\frac{41}{20}j}\|a\|_\infty^2 \omega(y)|x - x'|^{\frac{1}{20}}. \tag{3.6}$$

Now we denote $A' \triangle A = (A' \setminus A) \cup (A \setminus A')$. We claim that

$$\iint_{(A \triangle A') \setminus B(x, 2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}})} \frac{dudv}{|x_2 - v|} \leq C 2^{\frac{19}{20}j}|x - x'|^{\frac{1}{20}}. \tag{3.7}$$

Then by (3.7) we have

$$\left| \iint\limits_{(A\triangle A')\setminus B(x,2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}})} \psi(r)a(u,v)\omega(y+(r,0))\frac{dudv}{|x_2-v|}\right| \le C2^{-\frac{41}{20}j}\|a\|_\infty^2\omega(y)|x-x'|^{\frac{1}{20}}.$$

Now we come back to prove (3.7) first in the case $x_2'=x_2$. By similar triangles, we obtain that

$$|(A'\triangle A)\cap\{(a_1,a_2):a_2=v\}| \le \frac{2|x-x'||v-y_2|}{x_2-y_2}.$$

Then the integral in (3.7) has an estimate

$$\iint\limits_{(A\triangle A')\setminus B(x,2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}})} \frac{dudv}{|x_2-v|} \le \int_{y_2}^{x_2-c_0 2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}}} \frac{2|x-x'|}{x_2-v}\frac{v-y_2}{x_2-y_2}dv, \tag{3.8}$$

where c_0 is the minimum sine of the angle between vector $y+(2^{j-2},0)-x$ and $(1,0)$ and the angle between vector $y+(2^j,0)-x$ and $(1,0)$. By a geometry estimate we have $c_0\ge C\frac{|x_2-y_2|}{2^j}$. So (3.8) is controlled by

$$\frac{2|x-x'|}{|x_2-y_2|}\int_{c_0 2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}}}^{x_2-y_2} \frac{x_2-y_2-v}{v}dv \le C2^{\frac{19}{20}j}|x-x'|^{\frac{1}{20}}.$$

Now we consider the case where $x_2'>x_2$ or $x_2>x_2'$. By symmetry we only look at the case $x_2>x_2'>0$. We extend one of the sides of the shorter triangle A' to make it have the same height as A. Then we find a point x'' at the extend side such that $x_2=x_2''$. Since

$$|x''-x'|<C2^{\frac{1}{5}j}|x-x'|^{\frac{4}{5}},$$

then

$$|x-x''|\le C2^{\frac{1}{5}j}|x-x'|^{\frac{4}{5}}.$$

Replacing A' by the larger triangle A'' with the vertex $\{x'',y+(2^{j-2},0),y+(2^j,0)\}$, then $A\triangle A''$ contains $A\triangle A'$, and the ball $B(x,2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}})\supset B(x,C2^{\frac{11}{16}j}|x-x'|^{\frac{5}{16}})$ for some constant C. Therefore, we have

$$\iint_{(A\triangle A')\setminus B(x,2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}})}\frac{dudv}{|x_2-v|} \le \iint_{(A\triangle A'')\setminus B(x,c2^{\frac{11}{16}j}|x-x''|^{\frac{5}{16}})}\frac{dudv}{|x_2-v|}. \tag{3.9}$$

Use the same method as the case $x_2=x_2'$, we can get that the right side of (3.9) is bounded by $C2^{\frac{19}{20}j}|x-x'|^{\frac{1}{20}}$.

The remaining part is $(A\cap A')\setminus B(x,2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}})$. By straightforward computation we have

$$\begin{aligned} |r-r'| &\le |x_1-x_2| + \frac{|u-x_1'||x_2-x_2'||v-x_2'|}{|x_2-v||x_2'-v|} \\ &\quad + \frac{|x_1-x_1'||y_2-x_2||v-x_2'|+|u-x_1'||y_2-x_2'||x_2-x_2'|}{|x_2-v||x_2'-v|} \\ &\le C2^{\frac{19}{20}j}|x-x'|^{\frac{1}{20}}. \end{aligned} \tag{3.10}$$

Write

$$\iint_{A\cap A'\setminus B(x,2^{\frac{3}{4}j}|x-x'|^{\frac{1}{4}})}\left[\frac{\psi(r)a(u,v)\omega(y+r\theta)}{|x_2-v|}-\frac{\psi(r')a(u,v)\omega(y+r'\theta)}{|x_2'-v|}\right]dudv$$
$$=: G_1+G_2+G_3,$$

where

$$G_1=\iint(\psi(r)-\psi(r'))a(u,v)\omega(y+r\theta)\frac{dudv}{|x_2-v|},$$
$$G_2=\iint\psi(r')a(u,v)(\omega(y+r\theta)-\omega(y+r'\theta)\frac{dudv}{|x_2-v|},$$

and

$$G_3=\iint\psi(r')a(u,v)\omega(y+r'\theta)\left(\frac{1}{|x_2-v|}-\frac{1}{|x_2'-v|}\right)dudv.$$

For G_1, by (3.10) and $|z|^\alpha=|y+r\theta|^\alpha\le\omega(y)$, we have

$$|G_1|\le C2^{\frac{19}{20}j}|x-x'|^{\frac{1}{20}}\omega(y)\|a\|_\infty^2\frac{1}{2^{4j}}\iint\frac{dudv}{|x_2-v|}\le C2^{\frac{19}{20}j}|x-x'|^{\frac{1}{20}}\omega(y)\|a\|_\infty^2 2^{-3j}.$$

For G_2, by (3.10), $|y+r\theta|\ge C\cdot\max\{|y|,2^j\}$ and $|y+r'\theta|\ge C\cdot\max\{|y|,2^j\}$, we have

$$|G_2|\le C2^{\frac{19}{20}j}|x-x'|^{\frac{1}{20}}\|a\|_\infty^2 2^{-3j}\iiint_0^1|\nabla\omega(y+(sr+(1-s)r')\theta)|ds\frac{dudv}{|x_2-v|}$$
$$\le C2^{\frac{19}{20}j}|x-x'|^{\frac{1}{20}}2^{-3j}\|a\|_\infty^2\omega(y).$$

For G_3, we also get

$$|G_3|\le C2^{\frac{19}{20}j}|x-x'|^{\frac{1}{20}}2^{-3j}\|a\|_\infty^2\omega(y).$$

Combining above estimates, we have

$$|K_{j,j}(y,x)-K_{j,j}(y,x')|\le C2^{\frac{19}{20}j}|x-x'|^{\frac{1}{20}}2^{-3j}\|a\|_\infty^2\omega(y)$$

for any $|x-x'|\le 2^j\frac{1}{1000}$. Hence, we complete the proof of Lemma 3.1. □

Proof of Lemma 3.2 Fix $2^{j-3}\le|y|\le 2^{j+1}$. Let $K_j=K_j^1+K_j^2$, where

$$K_j^1(z-y)=K_j(z-y)\chi(z-y),$$

and χ is the characteristic function of the set $\{x:\rho(x,y)\le 2^{-\beta s}\}$ with $\rho(x,y)=|\frac{y}{|y|}-\frac{x}{|x|}|$ and β will be chosen later. Since $(T_j^*T_j)_\omega$ has kernel $K_{j,j}$, we write $K_{j,j}=K_{j,j}^1+K_{j,j}^2$, where

$$K_{j,j}^1(y,x)=\int K_j(z-x)K_j^1(z-y)m_{x,z}a\cdot m_{y,z}a\cdot\omega(z)dz$$

and

$$K_{j,j}^2(y,x)=\int K_j(z-x)K_j^2(z-y)m_{x,z}a\cdot m_{y,z}a\cdot\omega(z)dz.$$

Then we only need to prove Lemma 3.2 corresponding to $K_{j,j}^1$ and $K_{j,j}^2$. It is easy to check the term corresponding to $K_{j,j}^1$. Indeed, we have

$$|K_{j,j}^1(y,x)|\le C2^{-4j}\|a\|_\infty^2\int_{|z|\le C|y|}\chi(z)|z+y|^\alpha dz\le c2^{-2j}\|a\|_\infty^2|y|^\alpha 2^{-\beta s}, \tag{3.11}$$

where we use the following inequality (see (0.3) in [6, p. 425])

$$\int_0^{a|y|} \big|s\frac{x}{|x|}+y\big|^\alpha ds \le \begin{cases} C|y|^{\alpha+1}\big(1+|\frac{x}{|x|}+\frac{y}{|y|}|^{\alpha+1}\big) & \alpha\neq -1,\\ C\cdot \log^+|\frac{x}{|x|}+\frac{y}{|y|}| & \alpha=-1.\end{cases}$$

Then the corresponding term

$$\Big|\int K^1_{j,j}(y,x)B_{j-s}(x)dx\Big| \le C2^{-2j}\|a\|^2_\infty |y|^\alpha 2^{-\beta s}\int B_{j-s}(x)dx \le C\|a\|_\infty \lambda\omega(y)2^{-\beta s}. \tag{3.12}$$

Now we consider the remaining term corresponding to $K^2_{j,j}$. By the definition of χ, $\rho(y-z,y)\ge 2^{-\beta s}$. Since $\rho(y-z,y)=|\frac{y-z}{|y-z|}-\frac{y}{|y|}|\le 2\frac{|z|}{|y|}$, we have $|z|\ge C2^{-\beta s+j}$. Thus

$$\omega(z)\le C2^{-\alpha\beta s}\omega(y).$$

Applying the same method as proving Lemma 3.1 we may obtain

$$\Big|\int K^2_{j,j}(y,x)B_{j-s}(x)dx\Big| \le C\|a\|_\infty\lambda\omega(y)2^{\beta s(1-\alpha)-\frac{s}{5}}$$

as long as we choose $\beta=\frac{1}{10(1-\alpha)}$. □

3.2 Proof of Lemma 2.5

We write

$$\sum_{i\in\mathbb{Z}}\sum_{j\le i-3}\langle T_jB_{j-s},T_iB_{i-s}\rangle_\omega = \sum_{i\in\mathbb{Z}}\Big\langle \sum_{j\le i-3}(T_i^*T_j)_\omega B_{j-s},B_{i-s}\Big\rangle.$$

To prove Lemma 2.5, it is easy to see that it suffices to prove the following lemma:

Lemma 3.3. *For a fixed i, there exist C , $\varepsilon>0$ such that*

$$\Big|\sum_{j\le i-3}(T_i^*T_j)_\omega B_{j-s}(y)\Big| \le C2^{-\varepsilon s}\lambda\|a\|_\infty\omega(y)$$

for any $s\ge 10$.

Proof. The proof is similar to that of Lemma 2.4. So we only give the difference. Consider two cases of y as Lemma 3.1 and Lemma 3.2 respectively. We first consider the case $|y|<2^{j-3}$ or $|y|>2^{j+1}$. We proceed with the proof as we do in Lemma 3.1: For the analogous term of A_2, we switch to the polar coordinates $z=y+r\theta$

$$A_2=\int_{\mathbb{A}}\int_{2^{i-2}}^{2^i}\psi(r)(m_{x,y+r\theta}a-m_{x',y+r\theta})\omega(y+r\theta)drd\theta,$$

where

$$\psi(r)=K_j(y-x'+r\theta)K_i(r\theta)m_{y,y+r\theta}a\cdot r$$

and $\mathbb{A}$ is an arc in $\mathbb{S}^1$.

We claim that $\mathbb{A}$ is an arc of length of about 2^{-i+j}. Indeed, consider the support of K_j and K_i, we have

$$2^{j-2}\le|y-x'+r\theta|\le 2^j \qquad\text{and}\qquad 2^{i-2}\le r\le 2^i.$$

Since $j \le i-3$, we get $2^{i-3} \le |y-x'| \le 2^{i+1}$. Let Θ be the smallest cone with vertex at origin which contains the disc of radius 2^j at $y-x'$. Then the angle of Θ is at most a constant multiple of 2^{-i+j}. Since $|y-x'|-2^j \le r \le |y-x'|+2^j$, so the integrate area on r is

$$[|y-x'|-2^j, |y-x'|+2^j] \cap [2^{i-2}, 2^i]$$

and we set it as $[r_1, r_2]$. Using $j \le i-3$, we have the estimates $\|\psi\|_\infty \le C2^{-i}2^{-2j}\|a\|_\infty$ and $\|\psi'\|_\infty \le C2^{-i}2^{-3j}\|a\|_\infty$. After making a coordinate transform back, we get the integrate area A and A', where A is the triangle with vertices $\{y+(r_1,0), y+(r_2,0), x\}$ and A' is the triangle with vertices $\{y+(r_1,0), y+(r_2,0), x'\}$. Then we have

$$\int_A \frac{dudv}{|x_2-v|} = \int_{y_2}^{x_2} (r_2-r_1)\frac{x_2-v}{x_2-y_2}\frac{dv}{x_2-v} = r_2-r_1 \le 2^{j+1}.$$

Since we have assumed that $|x-x'| \le 2^j\frac{1}{1000}$, so it is not necessary to restrict $|x-y| > 10|x-x'|^{\frac{1}{10}}2^{\frac{9}{10}j}$ anymore. We just need $\frac{|x_2-y_2|}{|x-y|} > \frac{1}{10}t_0$ and $2^{i-3} \le |y-x| \le 2^{i+1}$. At last we obtain the Hölder smoothness estimate

$$|K_{i,j}(y,x) - K_{i,j}(y,x')| \le C2^{-2i}2^{-j/20}|x-x'|^{\frac{1}{20}}\omega(y). \tag{3.13}$$

Now we take a cube Q_n with side length 2^{j-s} and use the Hölder smoothness estimate (3.13) to get

$$\Big|\int K_{i,j}(y,x)b_n(x)dx\Big| = \Big|\int K_{i,j}(y,x) - K_{i,j}(y,x_{Q_n})b_n(x)dx\Big| \le C2^{-s/20}2^{-2i}\omega(y)\|a\|_\infty\|b_n\|_1$$

where x_{Q_n} is the center of Q_n. For a fixed y the kernel $K_{i,j}(y,x)$ is supported in $B(y,2^{i+1})$. Hence we have

$$\Big|\int K_{i,j}(y,x)B_{j-s}(x)dx\Big| \le C\|a\|_\infty^2 2^{-\frac{s}{20}}2^{-2i}\omega(y)\sum_{\substack{Q_n\in\mathfrak{Q}_{j-s}\\ Q_n\subset B(y,2^{i+1})}}\|b_n\|_1.$$

Then we sum over $j \le i-3$. Note that the cubes Q_n are disjoint each other, therefore we get

$$\sum_{j\le i-3}\sum_{\substack{Q_n\in\mathfrak{Q}_{j-s}\\ Q_n\subset B(y,2^{i+1})}}\|b_n\|_1 \le \sum_{j\le i-3}\sum_{\substack{Q_n\in\mathfrak{Q}_{j-s}\\ Q_n\subset B(y,2^{i+1})}}\frac{C\lambda|Q_n|}{\|a\|_\infty} \le \frac{C2^{2i}\lambda}{\|a\|_\infty}, \tag{3.14}$$

where the last inequality comes from all the cubes that appear in (3.14) are contained in a disc of radius 2^{i+1}. Hence we have

$$\Big|\sum_{j\le i-3}(T_i^*T_j)_\omega B_{j-s}(y)\Big| = \sum_{j\le i-3}\Big|\int K_{i,j}(y,x)B_{j-s}(x)dx\Big| \le C2^{-\varepsilon s}\lambda\|a\|_\infty\omega(y).$$

For the case $2^{j-3} < |y| < 2^{j+1}$. As in Lemma 3.2, we write $K_i = K_i^1 + K_i^2$ and denote

$$K_{i,j}^1 = \int K_j(z-x)K_i^1(z-y)m_{x,z}a\cdot\omega(z)dz$$

and

$$K_{i,j}^2 = \int K_j(z-x)K_i^2(z-y)m_{x,z}a\cdot\omega(z)dz.$$

A similar argument, which has been used to deal with the term corresponding to $K^1_{j,j}$ in proving Lemma 3.2, can be applied to estimate $K^1_{i,j}$. Combining the method we deal with $K^2_{j,j}$ in Lemma 3.2 and the method we handle $K_{i,j}$ in the case $|y| > 2^{j+1}$ or $|y| < 2^{j-3}$, we can get the proof of the term related to $K^2_{i,j}$. Therefore we complete the proof of Lemma 3.3. □

4. Proof of Theorem 1.2

In [7], Hofmann gave a weighted L^p boundedness for general singular integral operators. We will show that the commutator T_a discussed in this paper is an example of that general operator.

Before stating the theorem in [7], let us give some notations. For an open set Ω in $\mathbb{R}^d$, we denote by $C_c^\infty(\Omega)$ the set of functions with continuous derivatives of any order and compact support in Ω. Let $\psi \in C_c^\infty(\{|x| < 1\})$ be radial, non-trivial, have mean value zero, and be normalized so that $\int_0^\infty |\hat{\psi}(s)|^2 \frac{ds}{s} = 1$, then we define $Q_s f = \psi_s * f$, where $\psi_s(x) = s^{-d}\psi(\frac{x}{s})$.

Let $\mathfrak{D}$ denote the space of smooth function with compact support in $\mathbb{R}^d$ and $\mathfrak{D}'$ be its dual space. We assume that T maps $\mathfrak{D}$ to $\mathfrak{D}'$ and T is associated a kernel $K(x,y)$ in the sense that for $f, g \in C_c^\infty(\mathbb{R}^d)$ with disjoint support

$$\langle Tf, g\rangle = \iint K(x,y)f(y)g(x)dydx.$$

We will introduce some conditions similar to the conditions of $T1$ Theorem. We first suppose the kernel K satisfies the size condition:

$$|K(x,y)| \leq C_1|x-y|^{-d}. \tag{4.1}$$

Let $\varphi \in C_c^\infty(\frac{1}{2}, 2)$. Then for $v \in (0,\infty)$ we set

$$T_v f(x) = \int K(x,y)\varphi\Big(\frac{|x-y|}{v}\Big)f(y)dy.$$

We introduce the *weak smoothness condition* (WS):

$$\|Q_s T_v\|_{op} + \|Q_s T_v^*\|_{op} \leq C_2\|\psi\|_1(\|\varphi\|_\infty + \|\varphi'\|_\infty)(\frac{s}{v})^{\varepsilon_0}. \tag{4.2}$$

for some $0 < \varepsilon_0 \leq 1$ and $s < v$, where T^* is the adjoint operator of T and $\|\cdot\|_{op}$ denotes the norm of operator mapping L^2 to L^2.

As usual, we require the *weak boundedness property* (WBP):

$$\langle Th, \tilde{h}\rangle \leq C_3 R^d(\|h\|_\infty + R\|\nabla h\|_\infty)(\|\tilde{h}\|_\infty + R\|\nabla\tilde{h}\|_\infty). \tag{4.3}$$

for all $h, \tilde{h} \in C_c^\infty(\mathbb{R}^d)$ with support in any ball of radius R.

To define $T1$, we impose the *qualitative technical condition* (QT):

$$\begin{aligned}&\int_{|x-u|>2s}\Big|\int \psi_s(x-z)K(z,u)dz\Big|du < \infty,\\ &\int_{|x-u|>2s}\Big|\int \psi_s(x-z)K^*(z,u)dz\Big|du < \infty,\end{aligned} \tag{4.4}$$

where $K^*(z,u)$ is the kernel of T^*. Let $\tilde{\psi} \in C_c^\infty(B(x_0,s))$ and $\int \tilde{\psi}(x)dx = 0$. We write $1 = h + (1-h)$, where $h \in C_c^\infty(B(x_0,4s))$ and $h \equiv 1$ for $|x-x_0| \le 2s$. Then we define

$$\langle \tilde{\psi}, T1\rangle = \langle \tilde{\psi}, Th\rangle + \langle T^*\tilde{\psi}, 1-h\rangle,$$

where the second term is well defined by (4.4).

We consider truncations of T. Let $\Phi \in C_c^\infty(-1,1)$ and $\Phi \equiv 1$ on $[-\frac{1}{2},\frac{1}{2}]$. For $t < r$ and $f \in L^p$ with $1 < p < \infty$, $T_{(t,r)}$ is defined as follows:

$$T_{(t,r)}f(x) = \int K(x,y)\Big[\Phi\Big(\frac{|x-y|}{r}\Big) - \Phi\Big(\frac{|x-y|}{t}\Big)\Big]f(y)dy.$$

$T_{(0,r)}$ can be defined formally:

$$T_{(0,r)}f(x) = \int K(x,y)\Phi\Big(\frac{|x-y|}{r}\Big)f(y)dy.$$

Now we need two conditions to replace the usual condition $T1$, $T^*1 \in BMO$. One is the *quasi-Carleson measure condition* (QCM): For any ball B of radius $10\sqrt{d}t$, $1 < q < \infty$, $t > 0$, we have

$$\Big\|\Big(\int_0^t |Q_sT_{(0,t)}1|^2\frac{ds}{s}\Big)^{\frac{1}{2}}\Big\|_{L^q(B,\frac{dx}{t^d})} \le C_4, \tag{4.5}$$

where $\frac{dx}{t^d}$ denotes normalized Lebesgue measure.

The other is the *local paraproduct type condition* (LP): For all $r > 0$ and $1 < q < \infty$, for all $f \in C_c^\infty(\mathbb{R}^d)$ with support in any ball of radius $10\sqrt{d}r$, we have

$$\|\pi_r f\|_{L^q(B,\frac{dx}{r^d})} < C_5\|f\|_\infty, \tag{4.6}$$

where

$$\pi_r f = \int_0^r\int_0^t Q_s(Q_sT_{(t,r)}1Q_t^2f)\frac{ds}{s}\frac{dt}{t}.$$

Theorem A. (See [7, Theorem 2.14]) *Suppose that* T, T^* *and its kernel* K *satisfies* (4.1) $\sim$ (4.6). *Then for all* $\omega \in A_p$ *with* $1 < p < \infty$, *we have*

$$\|Tf\|_{p,\omega} \le C\Big(\sum_{1\le i\le 5} C_i\Big)\|f\|_{p,\omega} \tag{4.7}$$

To prove Theorem 1.2, we only need to verify that the commutators T_a, T_a^* and the kernel $L(x,y) = K(x-y)m_{x,y}(a)$ satisfies the conditions (4.1) $\sim$ (4.6) with C_i bounded by $C\|a\|_\infty$, $1 \le i \le 5$. It is trivial to see that $|L(x,y)| \le C\frac{\|a\|_\infty}{|x-y|^d}$. By the L^2 boundedness of T_a (see [2]), $\|T_a\|_{2\to2} \le C\|a\|_\infty$. Then we have

$$|\langle T_ah, \widetilde{h}\rangle| \le \|T_a\|_{2\to2}\|h\|_2\|\widetilde{h}\|_2 \le C\|a\|_\infty\|h\|_\infty\|\widetilde{h}\|_\infty R^d,$$

where $h, \widetilde{h} \in C_c^\infty(\mathbb{R}^d)$ and $h, \widetilde{h}$ support in $B(x_0,R)$. By Theorem A, the proof of the Theorem 1.2 follows immediately from the next four clams:

Clam 1: The operator T_a satisfies the weak smooth condition (4.2), which means that

$$\|Q_sT_{a,v}\|_{op} + \|Q_sT_{a,v}^*\|_{op} \le C\|a\|_\infty\|\psi\|_1(\|\varphi\|_\infty + \|\varphi'\|_\infty)(\frac{s}{v})^{\varepsilon_0},$$

where

$$T_{a,v}f(x) = \int L(x,y)\varphi\Big(\frac{|x-y|}{v}\Big)f(y)dy.$$

The proof of Clam 1 is similar to the proof of Lemma 4.3 in [7]. We just give the difference by the following lemma.

Lemma 4.1. *Let S_1 denote the convolution operator with the kernel $H(x) = K(x)\varphi(|x|)$, where K is a Calderón-Zygmund convolution kernel. Then $\|Q_s S_1\|_{op} \le Cs^{\varepsilon_0}$ for $s < 1$ and $0 < \varepsilon_0 < 1$.*

Proof. In fact, by Plancherel theorem we only need to check $|\hat{\psi}_s(\xi)\hat{H}(\xi)| \le Cs^{\varepsilon_0}$, for $0 < \varepsilon_0 < 1$. We firstly give an estimate of $\hat{H}(\xi)$. Write

$$\hat{H}(\xi) = \int_{S^{d-1}} \int e^{-2\pi i r\theta\cdot\xi} K(r\theta)\varphi(r)r^{d-1}drd\theta.$$

By Van der Corput's lemma, we have

$$\Big|\int e^{-2\pi i r\theta\cdot\xi} K(r\theta)\varphi(r)r^{d-1}dr\Big| \le \frac{C}{2\pi|\theta\cdot\xi|}.$$

On the other hand, by

$$\Big|\int e^{-2\pi i r\theta\cdot\xi} K(r\theta)\varphi(r)r^{d-1}dr\Big| \le C,$$

thus it is also dominated by $C|\theta\cdot\xi|^{-\varepsilon_0}$ for any $0 < \varepsilon_0 < 1$. So we have $|\hat{H}(\xi)| \le C|\xi|^{-\varepsilon_0}$. For $|s\xi| > 1$, $|\hat{\psi}_s(\xi)\hat{H}(\xi)| \le \|\psi\|_{L^1}|\xi|^{-\varepsilon_0} \le Cs^{\varepsilon_0}$. For $|s\xi| \le 1$, since

$$|\hat{\psi}_s(\xi)| = |\hat{\psi}(s\xi)| = \Big|\int (e^{-2\pi i s\xi x} - 1)\psi(x)dx\Big| \le C|s\xi|^{\varepsilon_0},$$

we have $|\hat{\psi}_s(\xi)\hat{H}(\xi)| \le Cs^{\varepsilon_0}$ for $0 < \varepsilon_0 < 1$. Hence we complete the proof. □

Clam 2: The operator T_a satisfies the technical condition (4.4).

Since $L^*(x,y)$ has the same form as $L(x,y)$, it is sufficient to prove (4.4) for $L(x,y)$. We need to use the following estimate (see Lemma 3 in [2, p. 68]):

$$\iiint_{\substack{x,y,y'\in B(x_0,R)\\ |y-y'|<r}} \big|(m_{x,y}a)^k - (m_{x,y'}a)^k\big|^2 dydy'dx \le Ck^2\Big(\frac{r}{R}\Big)^{\frac{2}{3}} r^d R^{2d}\|a\|_\infty^{2k}, \tag{4.8}$$

where $0 < r < R$, k is a positive integer. Since ψ has mean value zero, we have

$$\int_{|x-u|>2s} \Big|\int \psi_s(x-z)K(z-u)m_{z,u}adz\Big|du \le H_1 + H_2,$$

where

$$H_1 = \int_{|x-u|>2s} \Big|\int \psi_s(x-z)(K(z-u) - K(x-u))m_{z,u}adz\Big|du,$$

and

$$H_2 = \int_{|x-u|>2s} \Big|\int \psi_s(x-z)K(x-u)(m_{z,u}a - m_{x,u}a)dz|du.$$

For H_1, we have

$$H_1 \le \int_{|x-u|>2s}\int_{|x-z|<s} \frac{|z-x|}{s^d\cdot|x-u|^{d+1}}dzdu \le C.$$

For H_2, we have

$$H_2 = \sum_{j=0}^{+\infty} \int_{2^{j+1}s>|x-u|>2^j s} \Big| \int \psi_s(x-z)K(x-u)(m_{z,u}a - m_{x,u}a)dz \Big| du =: \sum_{j=0}^{+\infty} m_j(x).$$

Now we consider

$$\begin{aligned}
&\frac{1}{|B(x_0,2^j s)|}\int_{B(x_0,2^j s)} m_j(x)dx \\
&\leq \frac{1}{(2^j s)^d v_n}\int_{B(x_0,2^j s)}\int_{2^{j+1}s>|x-u|>2^j s}\int_{|x-z|<s}\frac{C}{s^d}\frac{1}{(2^j s)^d}|m_{z,u}a - m_{x,u}a|dzdudx \\
&\leq \frac{1}{(2^j s)^d v_n}\frac{C}{s^d(2^j s)^d}\iiint_{\substack{x,z,u\in B(x_0,2\cdot 2^j s)\\ |x-z|<s}} |m_{z,u}a - m_{x,u}a|dzdudx \\
&\leq C2^{-\frac{1}{3}j},
\end{aligned}$$

where the third inequality follows from Hölder's inequality and (4.8). Note that the constant C is independent of s, so $m_j(x) \leq Mm_j(x) \leq C\cdot 2^{-\frac{1}{3}j}$. We hence get

$$H_2 \leq \sum_{j=0}^{+\infty} C\cdot 2^{-\frac{1}{3}j} < C.$$

Clam 3: The operator T_a satisfies the quasi-Carleson measure condition (4.5).

By dilation invariance we may take $t=1$. Suppose B is a ball of radius $10\sqrt{d}$ with center x_0. We have

$$\langle T_{(0,1)}f, g\rangle = \iint K(x-y)m_{x,y}(a)\Phi(|x-y|)f(y)g(x)dydx.$$

Here and in the sequel we still use the notation $T_{(0,1)}$. Choose $\eta\in C_c^\infty(\mathbb{R}^d)$, such that $\eta(x)=1$ on $2B(x_0,10\sqrt{d})$ and $\eta(x)=0$ on $(4B(x_0,10\sqrt{d}))^c$. By the support of $Q_sT_{(0,1)}1$, we have

$$\Big(\int_B\Big(\int_0^1 |Q_sT_{(0,1)}1(x)|^2\frac{ds}{s}\Big)^{\frac{q}{2}}dx\Big)^{\frac{1}{q}} = \Big(\int_B\Big(\int_0^1 |Q_sT_{(0,1)}\eta(x)|^2\frac{ds}{s}\Big)^{\frac{q}{2}}dx\Big)^{\frac{1}{q}}. \tag{4.9}$$

By Littlewood-Paley theory (See [3]), (4.9) is majorized by $(\int_B(T_{(0,1)}\eta(x))^q dx)^{\frac{1}{q}}$. If the operator with kernel $K(\cdot)\Phi(|\cdot|)$ is bounded on L^2, then by Christ's result in [2], $\|T_{(0,1)}\|_{q\to q}\leq C\|a\|_\infty$ for all $1<q<+\infty$. Hence (4.9) is bounded.

Indeed, it is easy to check that $K(\cdot)\Phi(|\cdot|)$ is still a Calderón-Zygmund convolution kernel. Note that

$$\widehat{K\Phi}(\xi) = \hat{K}*\hat{\Phi}(\xi) = \int_{\mathbb{R}^n}\hat{K}(y)\hat{\Phi}(|\xi-y|)dy$$

is bounded since K is a Calderón-Zygmund convolution kernel and $\hat{\Phi}$ is a Schwartz function. So, the operator with the kernel $K(\cdot)\Phi(|\cdot|)$, initially defined on Schwartz class, has a bounded extension to an operator mapping $L^2(\mathbb{R}^n)$ to itself.

Clam 4: The operator T_a satisfies the local paraproduct condition (4.6).

By dilation invariance, we may take $r=1$. Let $f,g\in C_c^\infty(\mathbb{R}^d)$ with support in $B(x_0,10\sqrt{d})$. Fix $1<q<\infty$, by duality we only need to prove

$$\Big|\int_\varepsilon^1\int_\delta^t \langle Q_sT_{(t,1)}1Q_t^2f,Q_sg\rangle\frac{ds}{s}\frac{dt}{t}\Big|\le C\|a\|_\infty\|f\|_\infty\|g\|_{q'}. \tag{4.10}$$

We write

$$\langle Q_sT_{(t,1)}1Q_t^2f,Q_sg\rangle=\langle Q_tf,Q_t(Q_sT_{(t,1)}1Q_sg)\rangle.$$

Consider $Q_t(Q_sT_{(t,1)}1)(x)$, we can replace 1 by η, where $\eta\in C_c^\infty(\mathbb{R}^d)$ with $\eta\equiv1$ on $B(x,3t)$ and $\eta\equiv0$ on $(B(x,4t))^c$. By Hölder's inequality,

$$\begin{aligned}&|Q_t(Q_sT_{(t,1)}1Q_sg)(x)|\\ &\le C\Big(\frac{1}{t^n}\int_{|x-z|\le t}|Q_sT_{(t,1)}\eta(z)|^{q_1'}dz\Big)^{\frac{1}{q_1'}}\Big(\frac{1}{t^n}\int_{|x-z|\le t}|Q_sg(z)|^{q_1}dz\Big)^{\frac{1}{q_1}}\\ &\le C(M(|Q_sg|^{q_1})(x))^{\frac{1}{q_1}}\Big(\frac{1}{t^n}\int_{|x-z|\le t}|Q_sT_{(t,1)}\eta(z)|^{q_1'}dz\Big)^{\frac{1}{q_1'}},\end{aligned}\tag{4.11}$$

where we choose $1<q_1<2$. Since $\|Q_sT_{a,v}f\|_\infty\le C\|T_{a,v}f\|_\infty\le C\|a\|_\infty\|f\|_\infty$, by Clam 1 and using interpolation we have

$$\|Q_sT_{a,v}\|_{p\to p}\le C\|a\|_\infty\big(\frac{s}{v}\big)^{\varepsilon(p)} \tag{4.12}$$

for all $2\le p<+\infty$. Make a smooth partition of unity and write

$$\Phi\big(\frac{\rho}{t}\big)-\Phi(\rho)=\sum_{j:t\le 2^jt\le 4}\tilde\varphi\big(\frac{\rho}{2^jt}\big)\Big(\Phi\big(\frac{\rho}{t}\big)-\Phi(\rho)\Big),$$

where $\tilde\varphi\in C_c^\infty(\frac12,2)$ and $\sum_{j=-\infty}^\infty\tilde\varphi(\frac{\rho}{2^j})\equiv1$ for all $\rho>0$. We define

$$T_{a,j}f(x)=\int L(x,y)\tilde\varphi\Big(\frac{|x-y|}{2^jt}\Big)\Big(\Phi\big(\frac{|x-y|}{t}\big)-\Phi(x-y)\Big)f(y)dy,$$

then $T_{(t,1)}=\sum_{\frac{t}{4}\le 2^jt\le 2}T_{a,j}$. Applying Minkowski inequality and (4.12), we have

$$\begin{aligned}\Big(\frac{1}{t^n}\int_{|x-z|\le t}|Q_sT_{(t,1)}\eta(z)|^{q_1'}dz\Big)^{\frac{1}{q_1'}}&\le\Big(\frac{1}{t^d}\Big)^{\frac{1}{q_1'}}\sum_{\frac{t}{4}\le2^jt\le2}||Q_sT_{a,j}\eta||_{q_1'}\\ &\le\Big(\frac{1}{t^d}\Big)^{\frac{1}{q_1'}}\sum_{\frac{t}{4}\le2^jt\le2}C\|a\|_\infty\Big(\frac{s}{2^jt}\Big)^{\varepsilon(q_1')}\|\eta\|_{q_1'}\\ &\le C\|a\|_\infty\Big(\frac{s}{t}\Big)^{\varepsilon(q_1')}.\end{aligned}\tag{4.13}$$

By estimates (4.11), (4.13) and Hölder's inequality, the left side of (4.10) is bounded by

$$\begin{aligned}&\int_\varepsilon^1\int_\delta^t\Big(\frac{s}{t}\Big)^{\varepsilon(q_1')}\int_{\mathbb{R}^n}\big(M(|Q_sg|^{q_1})(x)\big)^{\frac{1}{q_1}}|Q_tf(x)|dx\frac{ds}{s}\frac{dt}{t}\\ &\le C\|a\|_\infty\int_\varepsilon^1\int_\delta^t\Big(\frac{s}{t}\Big)^{\varepsilon(q_1')}\|Q_sg\|_{2,\frac{1}{\omega}}\|Q_tf\|_{2,\omega}\frac{ds}{s}\frac{dt}{t}.\end{aligned}\tag{4.14}$$

By using Hölder's inequality again, the last term above is majorized by

$$C\|a\|_\infty\Big(\int_\varepsilon^1\int_\delta^t\Big(\frac{s}{t}\Big)^{\varepsilon(q_1')}\|Q_tf\|_{2,\omega}^2\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\Big(\int_\varepsilon^1\int_\delta^t\Big(\frac{s}{t}\Big)^{\varepsilon(q_1')}\|Q_tg\|_{2,\frac{1}{\omega}}^2\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}. \tag{4.15}$$

Firstly, let us consider the first integral factor in (4.15). Note that

$$\int_\delta^t\Big(\frac{s}{t}\Big)^{\varepsilon(q_1')}\frac{ds}{s}\le C,$$

hence by weighted Littlewood-Paley theory (See [8]), we have

$$\Big(\int_\varepsilon^1\int_\delta^t\Big(\frac{s}{t}\Big)^{\varepsilon(q_1')}\|Q_tf\|_{2,\omega}^2\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\le C\Big(\int_\varepsilon^1\|Q_tf\|_{2,\omega}^2\frac{dt}{t}\Big)^{\frac12}\le C\|f\|_{2,\omega}.$$

Then using the same method for the other factor, we get

$$\Big(\int_\delta^1\int_s^1\big(\frac{s}{t}\big)^{\varepsilon(q_1')}\|Q_sg\|_{2,\frac{1}{\omega}}^2\frac{dt}{t}\frac{ds}{s}\Big)^{\frac12}\le C\Big(\int_\delta^1\|Q_sg\|_{2,\frac{1}{\omega}}^2\frac{ds}{s}\Big)^{\frac12}\le C\|g\|_{2,\frac{1}{\omega}}.$$

Therefore (4.10) is controled by $\|a\|_\infty\|f\|_{2,\omega}\|g\|_{2,\frac{1}{\omega}}$. By extrapolation (See [4]), we can replace this bound by $\|a\|_\infty\|f\|_{q,\omega}\|g\|_{q',\frac{1}{\omega}}$, for any $1<q<\infty$. Now since ω is a Muckenhoupt weight, we can replace the bound by $\|a\|_\infty\|f\|_q\|g\|_{q'}$ by setting $\omega\equiv 1$. Since $f\in C_c^\infty(\mathbb{R}^d)$ with compact support, we have $\|f\|_q\le C\|f\|_\infty$. Hence we complete the proof of Theorem 1.2.

Acknowledgment. The authors would like to express their deep gratitude to the referee for his/her very careful reading, important comments and valuable suggestions.

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